Puzzle Corner
Taking Away by Ones and Twos
by Dave Youngs

This month's *Puzzle Corner* is designed to be done in conjunction with the *Maximizing Math* activity since both deal with the same mathematical game. The *Puzzle Corner* activity introduces students to the rules of the game and gives them the opportunity to play it a number of times. After students are familiar with the game, move on to the *Maximizing Math* activity which challenges them to examine the mathematics inherent within it.

Taking Away by Ones and Twos is similar to the historic nim* games which have their origins in ancient China. There are dozens of different versions of nim, but each involves two players who take turns removing objects from several separate piles with the winner being the person who takes the last object(s). (In a few versions, the *loser* is the one who takes the last object.) While *Taking Away by Ones and Twos* is related to the nim games, it differs in one significant way — the objects are all placed in only one pile. This difference makes the game easier to figure out from a mathematical standpoint and is, therefore, more appropriate for elementary students.

In *Taking Away by Ones and Twos*, two players start with a pile of 15 small objects, such as beans, pennies, or plastic chips. The players take turns removing either one or two objects from the pile on each move. The player to take the last one or two objects is the winner. Players alternate going first and should try to develop a winning strategy.

Once students have played the game enough to have learned some of its unwritten rules (such as, you will lose the game if there are three objects left when it's your turn), they are ready to begin the *Maximizing Math* activity which challenges them to look at the game from a mathematical standpoint. You will need to decide if you want to do the *Puzzle Corner* and the *Maximizing Math* activities on the same day or on separate days. My preference is to let students have several days to play the game before starting a formal exploration of its mathematics. This way, students are more likely to discover some of the underlying mathematics on their own through repeated playing of the game.

I hope that you and your students find this game enjoyable and rewarding. I'll have another activity for you next month.

Last Month's Puzzle

Stringing the Ring had two parts. The first part was designed to help students understand how the loop gets knotted around the ring in the magic trick.

The second part — applying this knowledge and performing the trick is a little more difficult. The secret is to get the ring to rotate when it is dropped so that the one edge of the ring goes into the open loop thus creating the knot. The amount of rotation is critical; if there is too little, the knot won't form and if there is too much, the ring tangles in the string. The amount of rotation necessary is slightly more than 90 degrees. Since the ring starts out horizontally this means it should rotate just past the vertical. There are several different methods to produce this rotation. The most elegant method I have seen was shown to me by Bill Jenner, an AIMS trainer from New York.

Bill had seen this trick being performed at a craft show and bought a setup from the performer for a few dollars. What Bill got was a brass bull's nose ring and a four foot loop of ball chain (like the ones used to turn on ceiling fans), but no instructions. After playing with this setup and trying various different techniques, this is what he found worked best for him:

> Bill places the ring around the loop. He lets one edge rest on top of his index finger and supports the opposite edge with his thumb (this support is at right angles to the plane formed by the loop). He then releases the ring with his thumb, letting it fall off the top of his index finger. This action causes the ring to tumble just the right amount and tie itself up into the chain. After a little practice, it works every time, Bill notes.

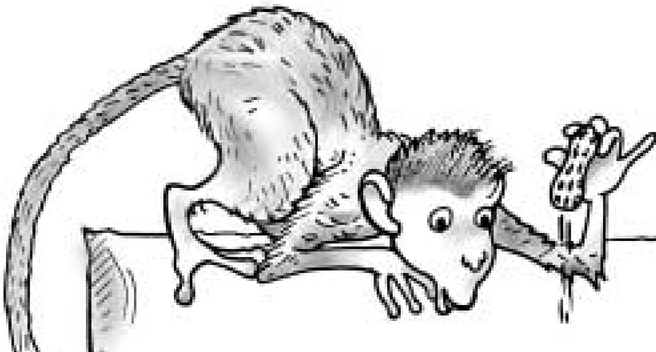
Another way to do the trick is to deliberately give the ring a twist when it is released. This technique takes more practice, and I have found that it isn't as consistent as Bill's method, since it is hard to get just the right amount of tumble to tie the knot.

A third method of doing this trick was shown to me recently by Joe Itel, an AIMS trainer from northern California. Joe is an amateur magician and often uses magic tricks in his math classes. He holds the ring by one edge using his thumb and index finger (this holding point is at right angles to the plane of the loop). Then he quickly flicks the ring into the opening of the loop. This action causes the ring to rotate as it goes through the loop thus tying the knot.

Both Bill and Joe have used this magic trick with their students. Bill's students were very interested in finding out just how the knot was tied and so they filmed the process with a video camera and then used the pause button on the VCR to see exactly what happens as the ring falls. Joe uses the trick to get his students to practice their written communication skills. After showing his students how to do this trick, he challenges his students to describe, in writing, exactly how to perform it. He notes that this is quite difficult to do and that it is a great exercise. Another way to extend this trick is to carefully test some of the variables — loop length, type of material for loop and ring, diameter and weight of ring, etc. — with the goal of finding the best way to perform the trick consistently. (The best set-up for doing this trick I have found so far is the brass ring and four-foot length of ball chain that Bill Jenner bought at the craft show.)

*Martin Gardner discusses the origin of the word *nim* in his book, *Wheels, Life and Other Mathematical Amusements*. He notes that Charles Bouton, a Harvard mathematician, first used this name in a 1901 article he published which analyzed the game. Gardner goes on to note that Bouton didn't explain why he chose the name, but conjectures that it could have come from the German word *nimm*, the imperative of *nehmen* (which means to take), or the archaic English word *nim* which meant to take and later became slang for stealing.

Taking Away *by* Ones *and* Twos



This is a game for two people. Place 15 small objects in a pile. Take turns removing either one or two objects from the pile. The winner is the player who picks up the last one or two objects.

Play this game several times. Be sure to take turns going first. Try to develop a winning strategy.

Maximizing Math

Taking Away by Ones and Twos: A Closer Look

by Dave Youngs

As mentioned in the *Puzzle Corner*, this *Maximizing Math* activity is designed to be done after students learn how to play the game introduced there. If your students haven't played the game yet, they will need to do so before starting this activity.

The game of *Taking Away by Ones and Twos* is based on simple arithmetic. Because of this, a definite winning strategy can be developed if some time and effort is expended in studying the underlying mathematics. Although understanding the mathematics involved in this game will lead to a winning strategy, the focus of this activity is on the mathematics. Therefore, students will need to work collaboratively instead of competitively.

As mentioned above, students should play the game a number of times before starting this activity. While playing, students may make some simple observations like the fact that they don't want three objects left on their turn or they will lose. At this point they are ready to stop playing and discuss the things they have learned about the game. It is important to communicate that the goal is not to keep a winning strategy secret, but to help everyone in the class figure out the mathematics on which that winning strategy is based.

While the mathematics in this game is fairly simple, it is worth exploring. As noted above, one of the first things students learn is that they will lose if there are three objects left when it is their turn. If they pick up one object, the other player picks up two and wins. If they pick up two objects the other player wins by picking up the last object. Thus, players begin to realize early on that three is an "unsafe" position. While this discovery becomes almost intuitive after playing a few games, students need to verbalize it.

The next discovery most students make is that six is also an unsafe number. If you are at six and take one object, your opponent can take two forcing you to the losing position of three. If you take two objects, your opponent takes one, again forcing you to three. Working backwards (an important problem-solving strategy) from six, students can discover that nine, 12, and 15 are also unsafe positions. This reasoning leads one to conclude that since the game begins with 15 objects, it is not desirable to go first. Conversely, the second player can always win if he or she is careful and avoids the unsafe positions.

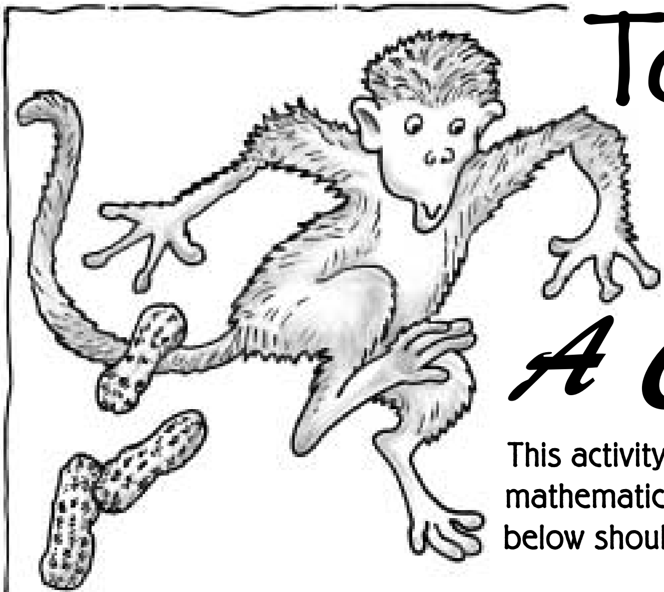
Once the above discoveries are made, some other interesting patterns emerge. Since each of the unsafe positions is a multiple of three and players are allowed to take either one or two objects per turn, one winning strategy is to go second and do the opposite of whatever the first player does. For example, if the first player takes one object, the second player takes two, forcing the first player to 12, an unsafe position. If the first player takes two objects next, the second takes one, forcing the first player to nine. This pattern continues until the second player wins the game.

Another pattern that can be discovered in this game is the alternating odd and even pattern for the multiples of three (three is odd, six is even, nine is odd, etc.). Students can also look at the patterns in the "safe" positions (14, 13, 11, 10, 8, 7, 5, 4, 2, 1) which have two consecutive numbers on either side of a multiple of three. With a little work, your students may discover some additional patterns.

If students accept the challenge to modify the game in some way, they will encounter a whole new set of patterns. For example, if players are allowed to pick up one, two, *or three* objects, the unsafe positions become four, eight, and 12 and the first player can always win (when starting with 15 objects) by taking three objects and then avoiding the unsafe positions. Another way to modify the game is to start with a different number of objects. Yet another modification is to place the objects in piles and allow objects to be taken from only one pile at a time. (This makes the game much closer to the historical nim games mentioned in the *Puzzle Corner*.) Hopefully students will experiment with several variations of the game and explore the mathematics in each game.

I hope that you find this game worthwhile and that your students discover some interesting mathematics while playing it. I will have another activity next month.

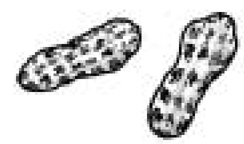
Taking Away *by* Ones *and* Twos: *A Closer Look*



This activity is designed to help you discover some of the mathematics in the game you have been playing. The questions below should help you do this.

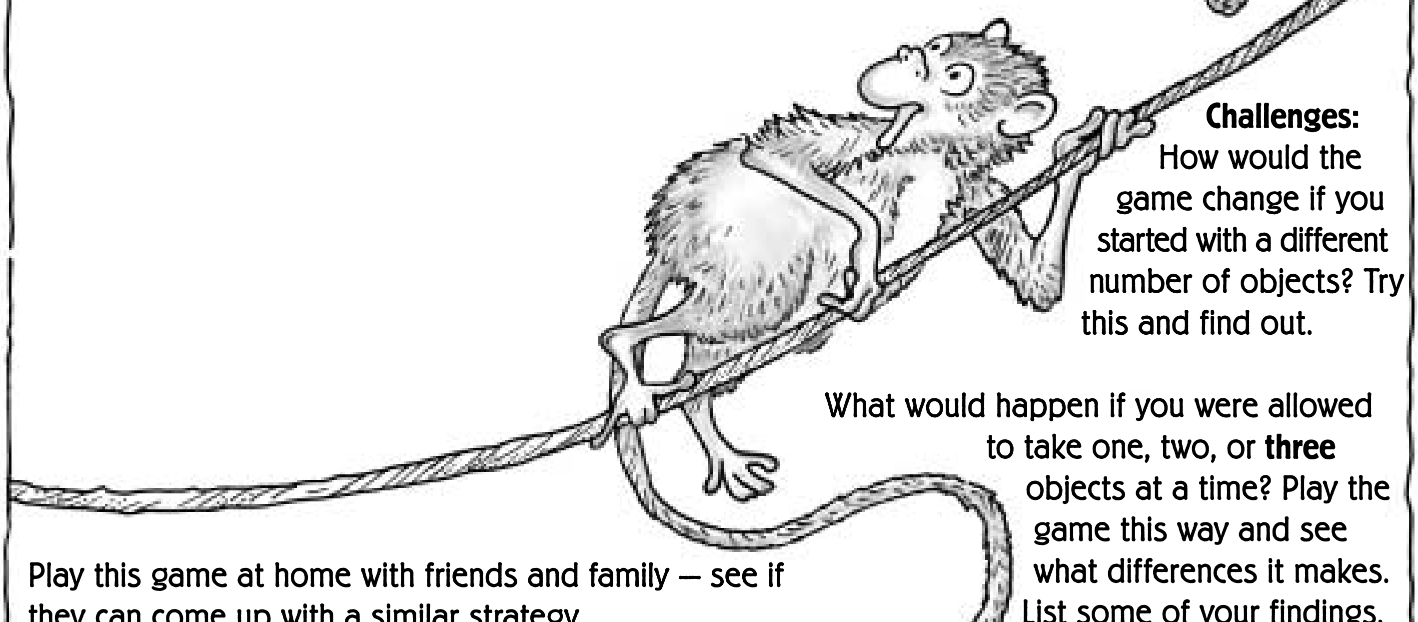
Does it make a difference who goes first? Why?

Are there some numbers of objects that you don't want to have when it's your turn? Why are these numbers problematic?



How would the game change if the person who takes the last object(s) loses?

Use what you have learned about this game to come up with a winning strategy. Describe your strategy using words and/or diagrams.



Challenges:
How would the game change if you started with a different number of objects? Try this and find out.

What would happen if you were allowed to take one, two, or **three** objects at a time? Play the game this way and see what differences it makes. List some of your findings.

Play this game at home with friends and family — see if they can come up with a similar strategy.